2.3 – Properties of Determinants; Cramer's Rule

Properties of Determinants

- For an $n \times n$ matrix A, $\det(kA) = k^n \det(A)$

- [**Lemma 2.3.2** (prelude to a later result)
  If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det(EB) = \det(E)\det(B)$.] (In brackets because it's here to support the next point)

- **Theorem 2.3.4** If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

- **Theorem 2.3.5** If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Theorem 2.3.3 A square matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem 2.3.8 Equivalent Statements (extends Theorem 1.6.4)
If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.
b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
g) $\det(A) \neq 0$.

16. Find the values of k for which the matrix A is invertible.

$A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$

35. In each part, find the determinant given that A is a 3×3 matrix for which $\det(A) = 7$.

a. $\det(3A)$

b. $\det(A^{-1})$

c. $\det(2A^{-1})$

d. $\det((2A)^{-1})$

Definition 1: If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the **matrix of cofactors from A**. The transpose of this matrix is called the **adjoint of A** and is denoted by $\text{adj}(A)$.

Theorem 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

20. Decide whether the matrix is invertible, and if so, use the adjoint to find its inverse.

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

Theorem 2.3.7 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}, \dots, \qquad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

26. Solve by Cramer's rule.

$$\begin{aligned} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$$